Lagrange linear partial differential equations

The equation of the form

$$Pp + Qq = R$$

is known as **Lagrange linear equation** and P, Q and R are functions of y and z. To solve this type of equations it is enough to solve the equation which the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

From the above subsidiary equation we can obtain two independent solutions $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, then the solution of the Lagrange's equation is given by $\phi(u, v) = 0$.

There are two methods of solving the subsidiary equation known as **method of grouping** and **method of multipliers**.

Method of Grouping

Consider the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Take any two ratios of the above equation say the first two or first and third or second and third. Suppose we take $\frac{dx}{P} = \frac{dy}{Q}$ and if the functions P and Q may contain the variable z, then eliminate the variable z. Then the direct integration gives $u(x, y) = c_1, v(y, z) = c_2$, then the solution of the Lagrange's equation is given by $\phi(u, v) = 0$.

Method of multipliers

Choose any three multipliers ℓ, m, n which may be constants or functions of x, y and z such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\ell dx + mdy + ndz}{\ell P + mQ + nR}.$$

If the relation $\ell P + mQ + nR = 0$, then $\ell dx + mdy + ndz$. Now direct integration gives us a solution

$$u(x, y, z) = c_1.$$

Similarly any other set of multipliers ℓ', m', n' gives another solution

$$v(x, y, z) = c_2.$$

Example 1.

Solve $xp + yq = z$.

Solution. *The subsidiary equation is* $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$. *Taking the first ratio we have* $\frac{dx}{x} = \frac{dy}{y}$. *Integrating we get*

$$\log \ x = \log \ y + \log c_1$$
$$\log \frac{x}{y} = \log \ c_1$$
$$\frac{x}{y} = c_1.$$

Taking the second and third ratios we have $\frac{dy}{y} = \frac{dz}{z}$. *Integrating we get*

$$\log y = \log \ z + \log \ c_2$$
$$\log \frac{y}{z} = \log c_2$$
$$\frac{y}{z} = c_2.$$

The required solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

Example 2.

Solve $xp + yq = x$.

Solution. *The subsidiary equation is* $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$. *Taking the first ratio we have* $\frac{dx}{x} = \frac{dy}{y}$. *Integrating we get*

$$\log\ x = \log\ y + \log\ c_1$$
$$\frac{x}{y} = c_1.$$

Taking the first and third ratios we have

$$\frac{dx}{x} = \frac{dz}{x}$$
$$dx = dz.$$

Integrating we get

$$x = z + c_2$$
$$x - z = c_2.$$

The required solution is $\phi\left(\frac{x}{y}, x - z\right) = 0$.

Example 3.

Solve $\tan x p + \tan y q = \tan z$.

Solution. *The subsidiary equation is* $\frac{dx}{\tan\ x} = \frac{dy}{\tan\ y} = \frac{dz}{\tan\ z}$.

Integrating $\frac{dx}{\tan\ x} = \frac{dy}{\tan\ y}$ *we get*

$$\log \sin x = \log \sin y + \log c_1 \implies \log \frac{\sin x}{\sin y} = \log c_1 \qquad \implies \frac{\sin x}{\sin y} = c_1$$

Integrating $\frac{dy}{\tan y} = \frac{dz}{\tan z}$ *we get*

$$\log \sin y = \log \sin y + \log c_2 \implies \log \frac{\sin\ y}{\sin z} = \log c_2 \qquad \implies \frac{\sin y}{\sin z} = c_2.$$

The required solution is $\phi\left(\frac{\sin\ x}{\sin\ y}, \frac{\sin y}{\sin\ z}\right) = 0$.

Example 4.

Find the complete integral of the partial differential equation $(1-x)p + (2-y)q = 3-z$.
Solution. *The subsidiary equation is*

$$\frac{dx}{1-x} = \frac{dy}{2-y} = \frac{dz}{3-z}.$$

Integrating $\frac{dx}{1-x} = \frac{dy}{2-y}$ *we get*

$$-\log(1-x) = -\log(2-y) + \log c_1 \implies \frac{2-y}{1-x} = c_1.$$

Integrating $\frac{dx}{1-x} = \frac{dz}{3-z}$ *we get*

$$-\log(1-x) = -\log(3-z) + \log c_2 \implies \frac{3-z}{1-x} = c_2.$$

The requird solution is $\phi\left(\frac{2-y}{1-x}, \frac{3-z}{1-x}\right) = 0$.

Example 5.

Solve $(y-z)p+(z-x)q=(x-y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}.$$

Using the multipliers $1,1,1$ *we have*

$$\textit{Each ratio} = \frac{dx+dy+dz}{y-z+z-x+x-y}=\frac{dx+dy+dz}{0} \implies x+y+z=c_1.$$

Using the multipliers x,y,z *we have*

$$\textit{Each ratio} = \frac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)}=\frac{xdx+ydy+zdz}{0} \implies x^2+y^2+z^2=2c_2.$$

Hence the solution is $\phi(x+y+z,x^2+y^2+z^2)=0$.

Example 6.

Solve $x(y-z)p + y(z-x)q = z(x-y)$.
Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}.$$

Using the multipliers $1, 1, 1$ *we have*

$$\text{Each ratio} = \frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz} = \frac{dx+dy+dz}{0} \implies x+y+z = c_1.$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z+z-x+x-y)} \implies \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \implies xyz = c_2.$$

Hence the solution is $\phi(x+y+z, xyz) = 0$.

Example 7.

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}.$$

Using the multipliers x, y, z *we have*

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + ydz}{0}$$
$$\implies x^2 + y^2 + z^2 = c_1.$$

Choosing the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \implies xyz = c_2.$$

The required solution is $\phi(x^2 + y^2 + z^2, xyz) = 0$.

Example 8.

Solve $x^2(y-z)+y^2(z-x)q=z^2(x-y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}.$$

Using the multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ *we have*

$$\text{Each ratio}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}\implies xyz=c_1.$$

Using the multipliers $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ *we have*

$$\text{Each ratio}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}\implies\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_2.$$

The required solution is $\phi(xyz,\frac{1}{x}+\frac{1}{y}+\frac{1}{z})=0$.

Example 9.

Solve $(4y-3z)p+(2z-4x)q=(3x-2y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{4y-3z}=\frac{dy}{2z-4x}=\frac{dz}{3z-2y}$. *Using the multipliers* $2,3,4$ *we have*

$$\text{Each ratio} = \frac{2dx+3dy+4dz}{2(4y-3z)+3(2z-4x)+4(3x-2y)} = \frac{2dx+3dy+4dz}{0}$$

$$\Rightarrow \quad 2dx+3dy+4dz=0 \implies 2x+3y+4z=0.$$

Using the multipliers x,y,z *we have*

$$\text{Each ratio} = \frac{xdx+ydy+zdz}{x(4y-3z)+y(2z-4x)+z(3x-2y)} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \quad xdx+ydy+zdz=0 \implies x^2+y^2+z^2=c_2.$$

The required solution $\phi(2x+3y+4z, x^2+y^2+z^2)=0$.

Example 10.

Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{x(y^2+z)} = \frac{dx}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$. *Using the multipliers* $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\text{Each ratio } = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-x^2-z+z^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0 \Longrightarrow \log\ x + \log\ y + \log\ z = \log\ c_1 \Longrightarrow xyz = c_1.$$

Using the multipliers $x, y, -1$ *we have*

$$\text{Each ratio} = \frac{xdx+ydy-dz}{z^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)} = \frac{xdx+ydy-dz}{x^2y^2+x^2z-y^2x^2-y^2z-zx^2+zy^2}$$

$$= \frac{xdx+ydy-dz}{0} \Rightarrow \quad xdx+ydy-dz = 0 \Longrightarrow x^2+y^2-2z = c_2.$$

The required solution is $\phi(xyz, x^2+y^2-2z) = 0$.

Example 11.

Find the general solution of $z(x-y) = x^2p - y^2q$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$. *Taking the first two ratios*

$$\frac{dx}{x^2} = \frac{dy}{-y^2} \implies -\frac{1}{x} = \frac{1}{y} + c_1 \implies \frac{1}{y} - \frac{1}{x} = c_1.$$

Adding first two ratios and comparing this with third

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{z(x-y)} \implies \frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{z(x-y)} \implies \frac{dx+dy}{x+y} = \frac{dz}{z}$$

$$\log(x+y) = \log z + \log c_2 \implies \log\frac{(x+y)}{z} = \log c_2 \implies \frac{x+y}{z} = c_2.$$

The required solution is $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{z+y}{z}\right) = 0$.

Example 12.

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{(x^2-y^2-z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$. *Taking the second and third ratios*

$$\frac{dy}{2xy} = \frac{dz}{2xz} \implies \frac{dy}{y} = \frac{dz}{z} \implies \log\ y = \log\ z + \log\ c_1 \implies \frac{y}{z} = c_1.$$

Using the multipliers x, y, z *we have*

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}.$$

Comparing this with the second ratio

$$\frac{dy}{2xy} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \implies \frac{dy}{y} = \frac{2(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)}$$

$$\log\ y = \log(x^2 + y^2 + z^2) + \log c_2 \implies \frac{y}{x^2 + y^2 + z^2} = c_2.$$

Hence the solution is $\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0$.

Example 13.

Solve $(x^2 - yz)p + (y^2 - xz)q = z^2 - xy$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - xz} = \frac{dz}{z^2 xy}.$$

Using the multipliers $1, 1, 1$ *we have*

$$\text{Each ratio } = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - xz - xy}. \tag{1}$$

Using the multipliers x, y, z *we have*

$$\text{Each ratio } = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}. \tag{2}$$

Solution (contd...)

Comparing (1) and (2) we have

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xz-xy} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xz-xy} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-xz-xy)}$$

$$dx+dy+dz = \frac{xdx+ydy+zdz}{(x+y+z)} \qquad \Longrightarrow \; xy+yz+xz = c_1.$$

Taking the first two ratios

$$\text{Each ratio } = \frac{dx-dy}{x^2-yz-(y^2-xz)} = \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dx-dy}{(x-y)(x+y+z)}. \tag{3}$$

Taking the second and third ratios

$$\text{Each ratio } = \frac{dy-dz}{y^2-xz-(z^2-xy)} = \frac{dy-dz}{y^2-z^2+x(y-z)} = \frac{dy-dz}{(y-z)(x+y+z)} \tag{4}$$

Comparing (3) and (4) we have

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} \Longrightarrow \frac{x-y}{y-z} = c_2.$$

Hence the solution is $\phi\left(xy+yz+xz, \frac{x-y}{y-z}\right) = 0.$

Example 14.

Solve $(x^2+y^2+yz)p+(x^2+y^2-xz)q=z(x+y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2+y^2+yz}=\frac{dy}{x^2+y^2-xz}=\frac{dz}{z(x+y)}.$$

Using the multipliers $1,-1,-1$ *we have*

$$\text{Each ratio}=\frac{dx-dy-dz}{x^2+y^2+yz-x^2-y^2+xz-zx-xy}=\frac{dx-dy-dz}{0}\implies x-y-z=c_1.$$

Using the multipliers $x,y,0$ *we have*

$$\text{Each ratio}=\frac{xdx+ydy}{x^3+xy^2+xyz+x^2y+y^3-xyz}=\frac{dz}{z(x+y)}$$

$$\frac{xdx+ydy}{(x+y)(x^2+y^2)}=\frac{dz}{z(x+y)}\implies\frac{xdx+ydy}{x^2+y^2}=\frac{dz}{z}\implies\frac{x^2+y^2}{z^2}=c_2.$$

Hence the solution is $\phi\left(x-y-z,\frac{x^2+y^2}{z^2}\right)=0$.

Example 15.

Solve $(x+y)zp + (x-y)zq = x^2 + y^2$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{(x+y)z} = \frac{dy}{(x-y)z} = \frac{dz}{x^2+y^2}.$$

Using the multipliers $x, -y, -z$ *we have*

$$\text{Each ratio } = \frac{xdx - ydy - zdz}{x^2z + xyz - xyz + y^2z - x^2z - y^2z} = \frac{xdx - ydy - zdz}{0}$$
$$\Rightarrow \quad xdx - ydy - zdz = 0 \implies x^2 - y^2 - z^2 = c_1.$$

Using the multipliers $y, x, -z$ *we have*

$$\text{Each ratio } = \frac{ydx + xdx - zdz}{xyz + y^2z + xz^2 - xyz - xz^2 - y^2z} = \frac{ydx + xdy - zdz}{0}$$
$$\implies ydx + xdx - zdz = 0 \implies 2xy - z^2 = c_2.$$

Hence the solution is $\phi(x^2 - y^2, z^2, 2xy - z^2) = 0$.